

# A sharp weighted anisotropic Poincaré inequality for convex domains

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We prove an optimal lower bound for the best constant in a class of weighted anisotropic Poincaré inequalities.

## 1 INTRODUCTION

In this paper we prove a sharp lower bound for the optimal constant  $\mu_{p,\mathcal{H},\omega}(\Omega)$  in the Poincaré-type inequality

$$\inf_{t \in \mathbb{R}} \|u - t\|_{L^p_\omega(\Omega)} \leq \frac{1}{[\mu_{p,\mathcal{H},\omega}(\Omega)]^{\frac{1}{p}}} \|\mathcal{H}(\nabla u)\|_{L^p_\omega(\Omega)},$$

with  $1 < p < +\infty$ ,  $\Omega$  is a bounded convex domain of  $\mathbb{R}^n$ ,  $\mathcal{H} \in \mathcal{H}(\mathbb{R}^n)$ , where  $\mathcal{H}(\mathbb{R}^n)$  is the set of lower semicontinuous functions, positive in  $\mathbb{R}^n \setminus \{0\}$  and positively 1-homogeneous; moreover, let  $\omega$  be a log-concave function.

If  $\mathcal{H}$  is the Euclidean norm of  $\mathbb{R}^n$  and  $\omega = 1$ , then  $\mu_p(\Omega) = \mu_{p,\varepsilon,\omega}(\Omega)$  is the first nontrivial eigenvalue of the Neumann  $p$ -Laplacian:

$$\begin{cases} -\Delta_p u = \mu_p |u|^{p-2} u & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, for a convex set  $\Omega$  it holds that

$$\mu_p(\Omega) \geq \left( \frac{\pi_p}{D_\varepsilon(\Omega)} \right)^p,$$

where

$$\pi_p = 2 \int_0^{+\infty} \frac{1}{1 + \frac{1}{p-1} s^p} ds = 2\pi \frac{(p-1)^{\frac{1}{p}}}{p \sin \frac{\pi}{p}}, \quad D_\varepsilon(\Omega) \text{ Euclidean diameter of } \Omega.$$

This estimate, proved in the case  $p = 2$  in [PW] (see also [B]), has been generalized the case  $p \neq 2$  in [AD, ENT, FNT, V] and for  $p \rightarrow \infty$  in [EKNT, RS]. Moreover the constant  $\left( \frac{\pi_p}{D_\varepsilon(\Omega)} \right)^p$  is the optimal constant of the one-dimensional Poincaré-Wirtinger inequality, with  $\omega = 1$ , on a segment of length  $D_\varepsilon(\Omega)$ . When  $p = 2$  and  $\omega = 1$ , in [BCDL] an extension of the estimate in the class of suitable non-convex domains has been proved.

The aim of the paper is to prove an analogous sharp lower bound for  $\mu_{p,\mathcal{H},\omega}(\Omega)$ , in a general anisotropic case. More precisely, our main result is:

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**Theorem 1.1.** Let  $\mathcal{H} \in \mathcal{H}(\mathbb{R}^n)$ ,  $\mathcal{H}^\circ$  be its polar function. Let us consider a bounded convex domain  $\Omega \subset \mathbb{R}^n$ ,  $1 < p < \infty$ , and take a positive log-concave function  $\omega$  defined in  $\Omega$ . Then, given

$$\mu_{p,\mathcal{H},\omega}(\Omega) = \inf_{\substack{u \in W^{1,\infty}(\Omega) \\ \int_{\Omega} |u|^{p-2} u \omega \, dx = 0}} \frac{\int_{\Omega} \mathcal{H}(\nabla u)^p \omega \, dx}{\int_{\Omega} |u|^p \omega \, dx},$$

it holds that

$$\mu_{p,\mathcal{H},\omega}(\Omega) \geq \left( \frac{\pi_p}{D_{\mathcal{H}}(\Omega)} \right)^p, \quad (1)$$

where  $D_{\mathcal{H}}(\Omega) = \sup_{x,y \in \Omega} \mathcal{H}^\circ(y - x)$ .

This result has been proved in the case  $p = 2$  and  $\omega = 1$ , when  $\mathcal{H}$  is a strongly convex, smooth norm of  $\mathbb{R}^n$  in [WX] with a completely different method than the one presented here.

In Section 2 below we give the precise definition of  $\mathcal{H}^\circ$  and give some details on the set  $\mathcal{H}(\mathbb{R}^n)$ . In Section 3 we give the proof of the main result.

## 2 NOTATION AND PRELIMINARIES

A function

$$\xi \in \mathbb{R}^n \mapsto \mathcal{H}(\xi) \in [0, +\infty[$$

belongs to the set  $\mathcal{H}(\mathbb{R}^n)$  if it verifies the following assumptions:

1.  $\mathcal{H}$  is positively 1-homogeneous, that is  
if  $\xi \in \mathbb{R}^n$  and  $t \geq 0$ , then  $\mathcal{H}(t\xi) = t\mathcal{H}(\xi)$ ;
2. if  $\xi \in \mathbb{R}^n \setminus \{0\}$ , then  $\mathcal{H}(\xi) > 0$ ;
3.  $\mathcal{H}$  is lower semi-continuous.

If  $\mathcal{H} \in \mathcal{H}(\mathbb{R}^n)$ , properties (1), (2), (3) give that there exists a positive constant  $\alpha$  such that

$$\alpha|\xi| \leq \mathcal{H}(\xi), \quad \xi \in \mathbb{R}^n.$$

The polar function  $\mathcal{H}^\circ: \mathbb{R}^n \rightarrow [0, +\infty[$  of  $\mathcal{H} \in \mathcal{H}(\mathbb{R}^n)$  is defined as

$$\mathcal{H}^\circ(\eta) = \sup_{\xi \neq 0} \frac{\langle \xi, \eta \rangle}{\mathcal{H}(\xi)}.$$

The function  $\mathcal{H}^\circ$  belongs to  $\mathcal{H}(\mathbb{R}^n)$ . Moreover it is convex on  $\mathbb{R}^n$ , and then continuous. If  $\mathcal{H}$  is convex, it holds that

$$\mathcal{H}(\xi) = (\mathcal{H}^\circ)^\circ(\xi) = \sup_{\eta \neq 0} \frac{\langle \xi, \eta \rangle}{\mathcal{H}^\circ(\eta)}.$$

If  $\mathcal{H}$  is convex and  $\mathcal{H}(\xi) = \mathcal{H}(-\xi)$  for all  $\xi \in \mathbb{R}^n$ , then  $\mathcal{H}$  is a norm on  $\mathbb{R}^n$ , and the same holds for  $\mathcal{H}^\circ$ .

We recall that if  $\mathcal{H}$  is a smooth norm of  $\mathbb{R}^n$  such that  $\nabla^2(\mathcal{H}^2)$  is positive definite on  $\mathbb{R}^n \setminus \{0\}$ , then  $\mathcal{H}$  is called a Finsler norm on  $\mathbb{R}^n$ .

If  $\mathcal{H} \in \mathcal{H}(\mathbb{R}^n)$ , by definition we have

$$\langle \xi, \eta \rangle \leq \mathcal{H}(\xi)\mathcal{H}^\circ(\eta), \quad \forall \xi, \eta \in \mathbb{R}^n. \quad (2)$$

**Remark 2.1.** Let  $\mathcal{H} \in \mathcal{H}(\mathbb{R}^n)$ , and consider the convex envelope of  $\mathcal{H}$ , that is the largest convex function  $\overline{\mathcal{H}}$  such that  $\overline{\mathcal{H}} \leq \mathcal{H}$ . It holds that  $\overline{\mathcal{H}}$  and  $\mathcal{H}$  have the same polar function:

$$(\overline{\mathcal{H}})^\circ = \mathcal{H}^\circ \quad \text{in } \mathbb{R}^n.$$

Indeed, being  $\overline{\mathcal{H}} \leq \mathcal{H}$ , by definition it holds that  $(\overline{\mathcal{H}})^\circ \geq \mathcal{H}^\circ$ . To show the reverse inequality, it is enough to prove that  $(\mathcal{H}^\circ)^\circ \leq \mathcal{H}$ . Then, being  $\overline{\mathcal{H}}$  the convex envelope of  $\mathcal{H}$ , it must be  $(\mathcal{H}^\circ)^\circ \leq \overline{\mathcal{H}}$ , that implies  $(\overline{\mathcal{H}})^\circ \leq \mathcal{H}^\circ$ . Denoting by  $G(x) = (\mathcal{H}^\circ)^\circ(x)$ , for any  $x$  there exists  $\bar{v}_x$  such that

$$G(x) = \frac{\langle x, \bar{v}_x \rangle}{\mathcal{H}^\circ(\bar{v}_x)}, \quad \text{and} \quad \langle x, \bar{v}_x \rangle \leq \mathcal{H}^\circ(\bar{v}_x) \mathcal{H}(x), \quad \text{that implies} \quad G(x) \leq \mathcal{H}(x).$$

Let  $\mathcal{H} \in \mathcal{H}(\mathbb{R}^n)$ , and consider a bounded convex domain  $\Omega$  of  $\mathbb{R}^n$ . Throughout the paper  $D_{\mathcal{H}}(\Omega) \in ]0, +\infty[$  will be

$$D_{\mathcal{H}}(\Omega) = \sup_{x, y \in \Omega} \mathcal{H}^\circ(y - x).$$

We explicitly observe that since  $\mathcal{H}^\circ$  is not necessarily even, in general  $\mathcal{H}^\circ(y - x) \neq \mathcal{H}^\circ(x - y)$ . When  $\mathcal{H}$  is a norm, then  $D_{\mathcal{H}}(\Omega)$  is the so called anisotropic diameter of  $\Omega$  with respect to  $\mathcal{H}^\circ$ . In particular, if  $\mathcal{H} = \mathcal{E}$  is the Euclidean norm in  $\mathbb{R}^n$ , then  $\mathcal{E}^\circ = \mathcal{E}$  and  $D_{\mathcal{E}}(\Omega)$  is the standard Euclidean diameter of  $\Omega$ . We refer the reader, for example, to [CS, FFK] for remarkable examples of convex not even functions in  $\mathcal{H}(\mathbb{R}^n)$ . On the other hand, in [VS] some results on isoperimetric and optimal Hardy-Sobolev inequalities for a general function  $\mathcal{H} \in \mathcal{H}(\mathbb{R}^n)$  have been proved, by using a generalization of the so called convex symmetrization introduced in [AFLT] (see also [DG1, DG2, DG3]).

**Remark 2.2.** In general  $\mathcal{H}$  and  $\mathcal{H}^\circ$  are not rotational invariant. Anyway, if  $A \in SO(n)$ , defining

$$\mathcal{H}_A(x) = \mathcal{H}(Ax), \tag{3}$$

and being  $A^T = A^{-1}$ , then  $\mathcal{H}_A \in \mathcal{H}(\mathbb{R}^n)$  and

$$(\mathcal{H}_A)^\circ(\xi) = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\langle x, \xi \rangle}{\mathcal{H}_A(x)} = \sup_{y \in \mathbb{R}^n \setminus \{0\}} \frac{\langle A^T y, \xi \rangle}{\mathcal{H}(y)} = \sup_{y \in \mathbb{R}^n \setminus \{0\}} \frac{\langle y, A\xi \rangle}{\mathcal{H}(y)} = (\mathcal{H}^\circ)_A(\xi).$$

Moreover,

$$D_{\mathcal{H}_A}(A^T \Omega) = \sup_{x, y \in A^T \Omega} (\mathcal{H}^\circ)_A(y - x) = \sup_{\bar{x}, \bar{y} \in \Omega} \mathcal{H}^\circ(\bar{y} - \bar{x}) = D_{\mathcal{H}}(\Omega). \tag{4}$$

### 3 PROOF OF THE PAYNE-WEINBERGER INEQUALITY

In this section we state and prove Theorem 1.1. To this aim, the following Wirtinger-type inequality, contained in [FNT] is needed.

**Proposition 3.1.** *Let  $f$  be a positive log-concave function defined on  $[0, L]$  and  $p > 1$ , then*

$$\inf \left\{ \frac{\int_0^L |u'|^p f \, dx}{\int_0^L |u|^p f \, dx}, u \in W^{1,p}(0, L), \int_0^L |u|^{p-2} u f \, dx = 0 \right\} \geq \frac{\pi_p^p}{L^p}.$$

The proof of the main result is based on a slicing method introduced in [PW] in the Laplacian case. The key ingredient is the following Lemma. For a proof, we refer the reader, for example, to [PW, B, FNT].

**Lemma 3.2.** *Let  $\Omega$  be a convex set in  $\mathbb{R}^n$  having (Euclidean) diameter  $D_{\mathcal{E}}(\Omega)$ , let  $\omega$  be a positive log-concave function on  $\Omega$ , and let  $u$  be any function such that  $\int_{\Omega} |u|^{p-2} u \omega \, dx = 0$ . Then, for all positive  $\varepsilon$ , there exists a decomposition of the set  $\Omega$  in mutually disjoint convex sets  $\Omega_i$  ( $i = 1, \dots, k$ ) such that*

$$\bigcup_{i=1}^k \overline{\Omega_i} = \overline{\Omega}$$

$$\int_{\Omega_i} |u|^{p-2} u \omega \, dx = 0$$

and for each  $i$  there exists a rectangular system of coordinates such that

$$\Omega_i \subset \{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq d_i, |x_l| \leq \varepsilon, l = 2, \dots, n\},$$

where  $d_i \leq D_{\mathcal{E}}(\Omega)$ ,  $i = 1, \dots, k$ .

**Proof of Theorem 1.1.** By density, it is sufficient to consider a smooth function  $u$  with uniformly continuous first derivatives and  $\int_{\Omega} |u|^{p-2} u \omega \, dx = 0$ .

Hence, we can decompose the set  $\Omega$  in  $k$  convex domains  $\Omega_i$  as in Lemma 3.2. In order to prove (1), we will show that for any  $i \in \{1, \dots, k\}$  it holds that

$$\int_{\Omega_i} H^p(\nabla u) \omega \, dx \geq \frac{\pi_p^p}{D_{\mathcal{H}}(\Omega)^p} \int_{\Omega_i} |u|^p \omega \, dx. \quad (5)$$

By Lemma 3.2, for each fixed  $i \in \{1, \dots, k\}$ , there exists a rotation  $A_i \in SO(n)$  such that

$$A_i \Omega_i \subset \{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq d_i, |x_l| \leq \varepsilon, l = 2, \dots, n\}.$$

By changing the variable  $y = A_i x$ , recalling the notation (3) and using (4) it holds that

$$\int_{\Omega_i} \mathcal{H}^p(\nabla u(x)) \omega(x) \, dx = \int_{A_i \Omega_i} \mathcal{H}_{A_i^T}(\nabla u(A_i^T y))^p \omega(A_i^T y) \, dy; \quad D_{\mathcal{H}}(\Omega) = D_{\mathcal{H}_{A_i^T}}(A_i \Omega).$$

We deduce that it is not restrictive to suppose that for any  $i \in \{1, \dots, n\}$   $A_i$  is the identity matrix, and the decomposition holds with respect to the  $x_1$ -axis.

Now we may argue as in [FNT]. For any  $t \in [0, d_i]$  let us denote by  $v(t) = u(t, 0, \dots, 0)$ , and  $f_i(t) = g_i(t) \omega(t, 0, \dots, 0)$ , where  $g_i(t)$  will be the  $(n-1)$  volume of the intersection of  $\Omega_i$  with the hyperplane  $x_1 = t$ . By Brunn-Minkowski inequality  $g_i$ , and then  $f_i$ , is a log-concave function in  $[0, d_i]$ . Since  $u, u_{x_1}$  and  $\omega$  are uniformly continuous in  $\Omega$  there exists a modulus of continuity  $\eta(\cdot)$  with  $\eta(\varepsilon) \searrow 0$  for  $\varepsilon \rightarrow 0$ , independent of the decomposition of  $\Omega$  and such that

$$\left| \int_{\Omega_i} |u_{x_1}|^p \omega \, dx - \int_0^{d_i} |v'|^p f_i \, dt \right| \leq \eta(\varepsilon) |\Omega_i|, \quad \left| \int_{\Omega_i} |u|^p \omega \, dx - \int_0^{d_i} |v|^p f_i \, dt \right| \leq \eta(\varepsilon) |\Omega_i|,$$

and

$$\left| \int_0^{d_i} |v|^{p-2} v f_i \, dt \right| \leq \eta(\varepsilon) |\Omega_i|.$$

Now, by property (2) we deduce that for any vector  $\eta \in \mathbb{R}^n$

$$|\langle \nabla u, \eta \rangle| \leq \mathcal{H}(\nabla u) \max\{\mathcal{H}^0(\eta), \mathcal{H}^0(-\eta)\}.$$

Then choosing  $\eta = e_1$  and denoting by  $M = \max\{\mathcal{H}^o(e_1), \mathcal{H}^o(-e_1)\}$ , Proposition 3.1 gives

$$\begin{aligned} \int_{\Omega_i} \mathcal{H}^p(\nabla u) \omega \, dx &\geq \frac{1}{M^p} \int_{\Omega_i} |u_{x_1}|^p \omega \, dx \geq \frac{1}{M^p} \int_0^{d_i} |v'|^p f_i \, dt - \frac{\eta(\varepsilon)|\Omega_i|}{M^p} \\ &\geq \frac{\pi_p}{d_i^p M^p} \int_0^{d_i} |v|^p f_i \, dt + C\eta(\varepsilon)|\Omega_i| \geq \frac{\pi_p^p}{d_i^p M^p} \int_{\Omega_i} |u|^p \omega \, dx + C\eta(\varepsilon)|\Omega_i|, \end{aligned}$$

where  $C$  is a constant which does not depend on  $\varepsilon$ . Being  $d_i \leq D_\varepsilon(\Omega)$ , and then  $d_i M \leq D_{\mathcal{H}}(\Omega)$ , by letting  $\varepsilon$  to zero we get (5). Hence, by summing over  $i$  we get the thesis.

**Remark 3.3.** In order to prove an estimate for  $\mu_{p,\mathcal{H},\omega}$ , we could use directly property (2) with  $v = \frac{\nabla u}{|\nabla u|}$ , and the Payne-Weinberger inequality in the Euclidean case, obtaining that

$$\int_{\Omega} \mathcal{H}^p(\nabla u) \omega \, dx \geq \int_{\Omega} \frac{|\nabla u|^p}{\mathcal{H}^o(v)^p} \omega \, dx \geq \frac{\pi_p^p}{D_\varepsilon(\Omega)^p \mathcal{H}^o(v_m)^p} \int_{\Omega} |u|^p \omega \, dx,$$

where  $\mathcal{H}^o(v_m) = \max_{|v|=1} \mathcal{H}^o(v)$ . However, we have a worst estimate than (1) because  $D_\varepsilon(\Omega) \cdot \mathcal{H}^o(v_m)$  is, in general, strictly larger than  $D_{\mathcal{H}}(\Omega)$ , as shown in the following example.

**Example 1.** Let  $\mathcal{H}(x, y) = \sqrt{a^2 x^2 + b^2 y^2}$ , with  $a < b$ . Then  $\mathcal{H}$  is a even, smooth norm with  $\mathcal{H}^o(x, y) = \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$  and the Wulff shapes  $\{\mathcal{H}^o(x, y) < R\}$ ,  $R > 0$ , are ellipses. Clearly we have:

$$D_\varepsilon(\Omega) = 2b \quad \text{and} \quad D_{\mathcal{H}}(\Omega) = 2$$

Let us compute  $\mathcal{H}^o(v_m)$ . We have:

$$\max_{|v|=1} \mathcal{H}^o(v) = \max_{\vartheta \in [0, 2\pi]} \sqrt{\frac{(\cos \vartheta)^2}{a^2} + \frac{(\sin \vartheta)^2}{b^2}} = \mathcal{H}^o(0, \pm 1) = \frac{1}{a}.$$

Then  $D_\varepsilon(\Omega) \cdot \mathcal{H}^o(v_m) = 2\frac{b}{a} > 2$ .

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